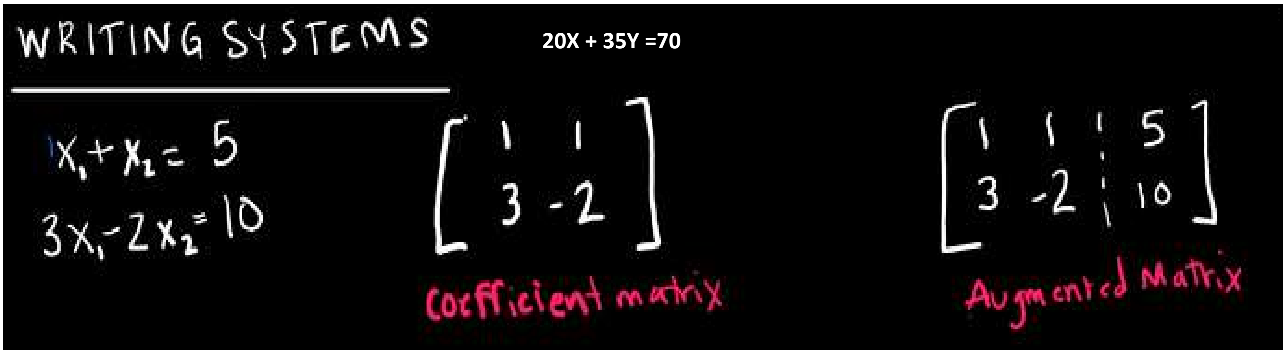
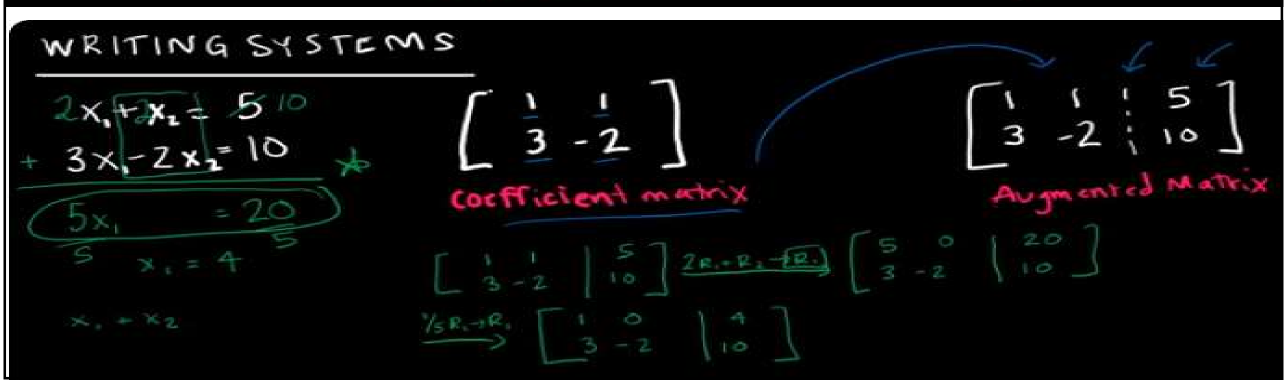
ENG 301 Engineering Mathematics III

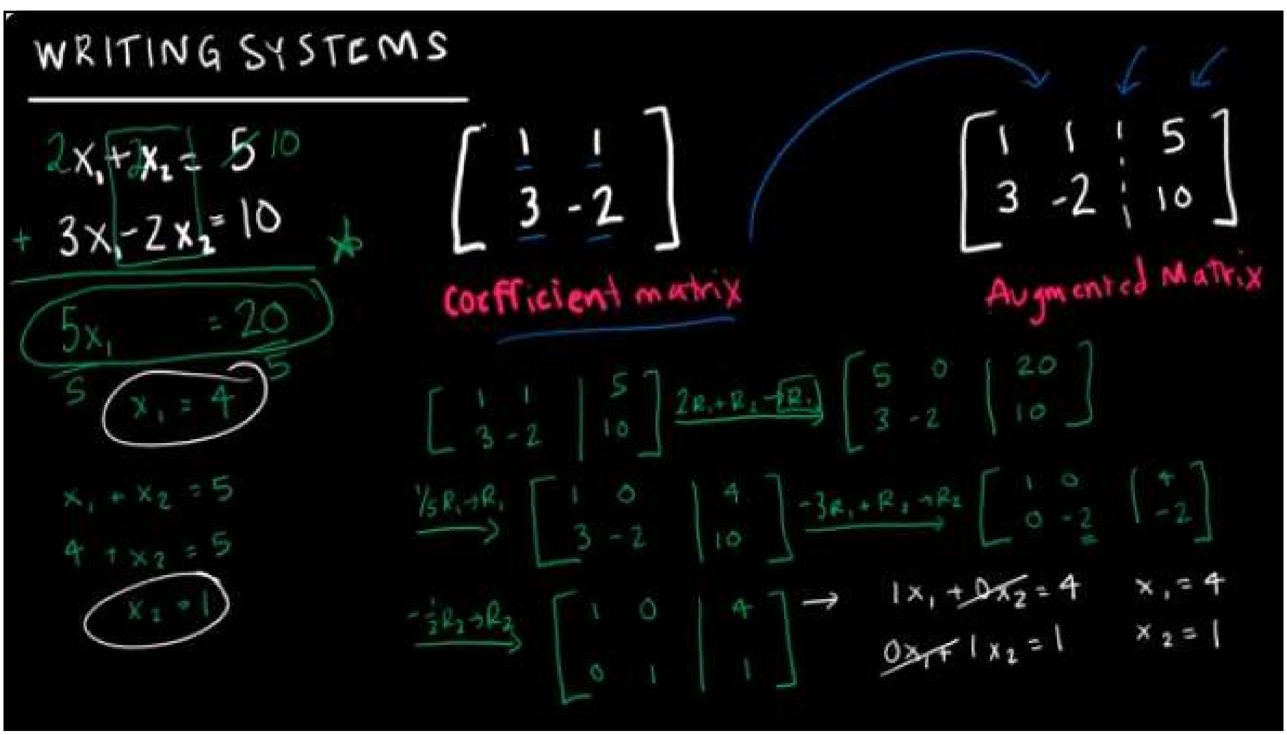
Lecture by: Engr. O. D. Adigun

LINEAR ALGEBRA

SYSTEMS OF LINEAR EQUATIONS


WRITING SYSTEMS
20X + 35Y =70
1x1 + x2 = 5
3x1 - 2x2 = 10
1 1
3 -2
Coefficient matrix
1 1 | 5
3 -2 | 10
Augmented Matrix


WRITING SYSTEMS
2x1 + 2x2 = 5 10
+ 3x1 - 2x2 = 10
5x1 = 20
x1 = 4
x1 + x2
Coefficient matrix
Augmented Matrix
2R1 + R2 → R1
1/5 R1 → R1


WRITING SYSTEMS
2x1 + 2x2 = 5 10
+ 3x1 - 2x2 = 10
5x1 = 20
x1 = 4
x1 + x2 = 5
4 + x2 = 5
x2 = 1
Coefficient matrix
Augmented Matrix
2R1 + R2 → R1
1/5 R1 → R1
-3R1 + R2 → R2
-1/2 R2 → R2
1x1 + 0x2 = 4
0x1 + 1x2 = 1
x1 = 4
x2 = 1

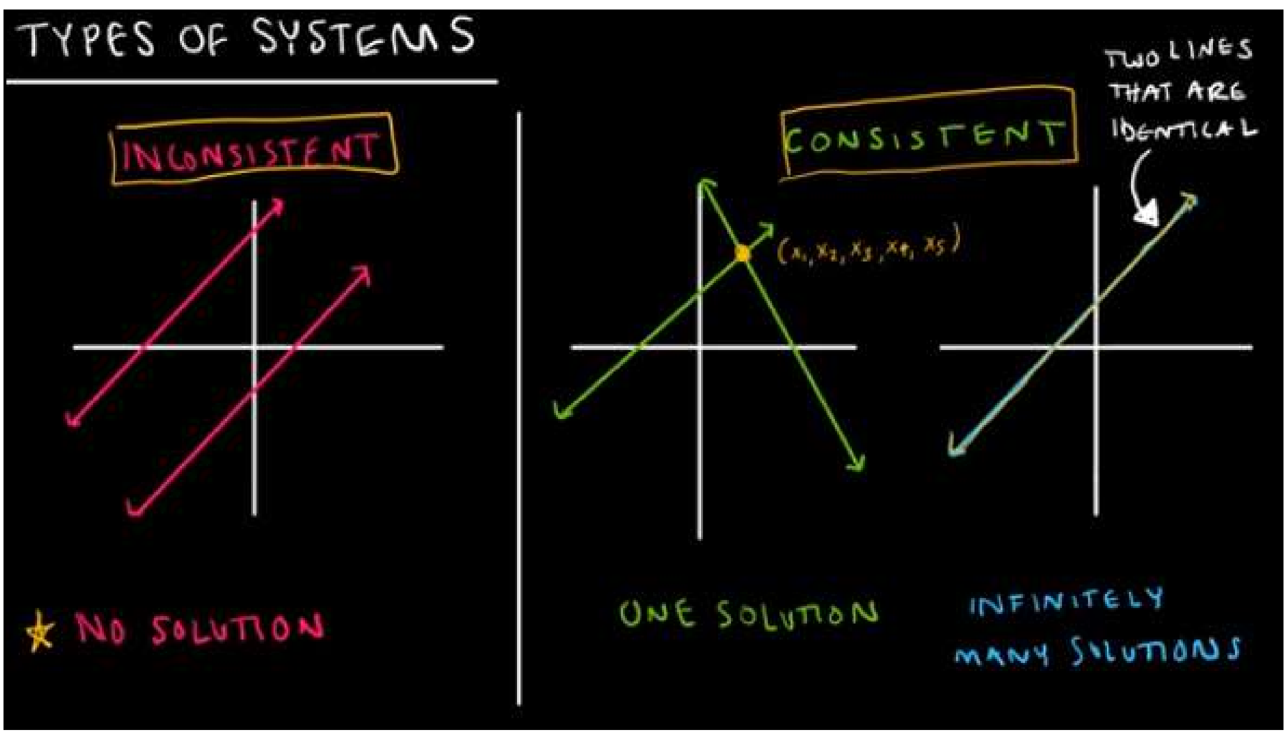
TERMINOLOGY

LINEAR EQUATION - An equation that can be written as $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$ where $a_1, a_2, \dots a_n, b$ are real or complex numbers.

SYSTEM OF LINEAR EQUATIONS - A collection of two or more linear equations using the same variables.

SOLUTION - A list of numbers $(s_1, s_2, s_3 \dots)$ that makes each equation in the system true when substituted for $x_1, x_2, x_3 \dots$ respectively.

SOLUTION SET - The set of all possible solutions to a system.



PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION.

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

$(29, 16, 3)$

PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION.

$$4(x_1 - 2x_2 + x_3 = 0)$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$2x_2 - 8x_3 = 8$$

$$4x_1 - 8x_2 + 4_3 = 0$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION.

$$4(x_1 - 2x_2 + x_3 = 0)$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$2x_2 - 8x_3 = 8$$
$$-3x_2 + 13x_3 = -9$$

$$4x_1 - 8x_2 + 4_3 = 0$$
$$+ \quad -x_1 + 5x_2 + 9x_3 = -9$$

PRACTICE

SOLVE THE GIVEN SYSTEM OF EQUATIONS USING ELIMINATION.

$$4(x_1 - 2x_2 + x_3 = 0)$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$3(2x_2 - 8x_3 = 8) \rightarrow 6x_2 - 24x_3 = 24$$
$$2(-3x_2 + 13x_3 = -9) \rightarrow + \; -6x_2 + 26x_3 = -18$$

$$2x_3 = 6$$
$$x_3 = 3$$

$$2x_2 - 8x_3 = 8$$
$$2x_2 - 8(3) = 8$$
$$2x_2 - 24 = 8$$
$$2x_2 = 32$$
$$x_2 = 16$$

$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 - 2(16) + 3 = 0$$
$$x_1 - 32 + 3 = 0$$
$$x_1 - 29 = 0$$
$$x_1 = 29$$

$$\boxed{(29, 16, 3)}$$

LINEAR ALGEBRA

SOLVE SYSTEMS USING AUGMENTED MATRICES AND ROW OPERATIONS

ROW OPERATIONS

REPLACEMENT - REPLACE ONE ROW BY THE SUM OF ITSELF AND A MULTIPLE OF ANOTHER ROW

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right]$$

INTERCHANGE - INTERCHANGE (SWAP) TWO ROWS

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right]$$

SCALING - MULTIPLY A ROW BY A NON-ZERO CONSTANT

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right]$$

ROW OPERATIONS

REPLACEMENT - REPLACE ONE ROW BY THE SUM OF ITSELF AND A MULTIPLE OF ANOTHER ROW

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_2} \left[\begin{array}{cc|c} 2 & 4 & 8 \\ 0 & 4 & -10 \end{array}\right]$$

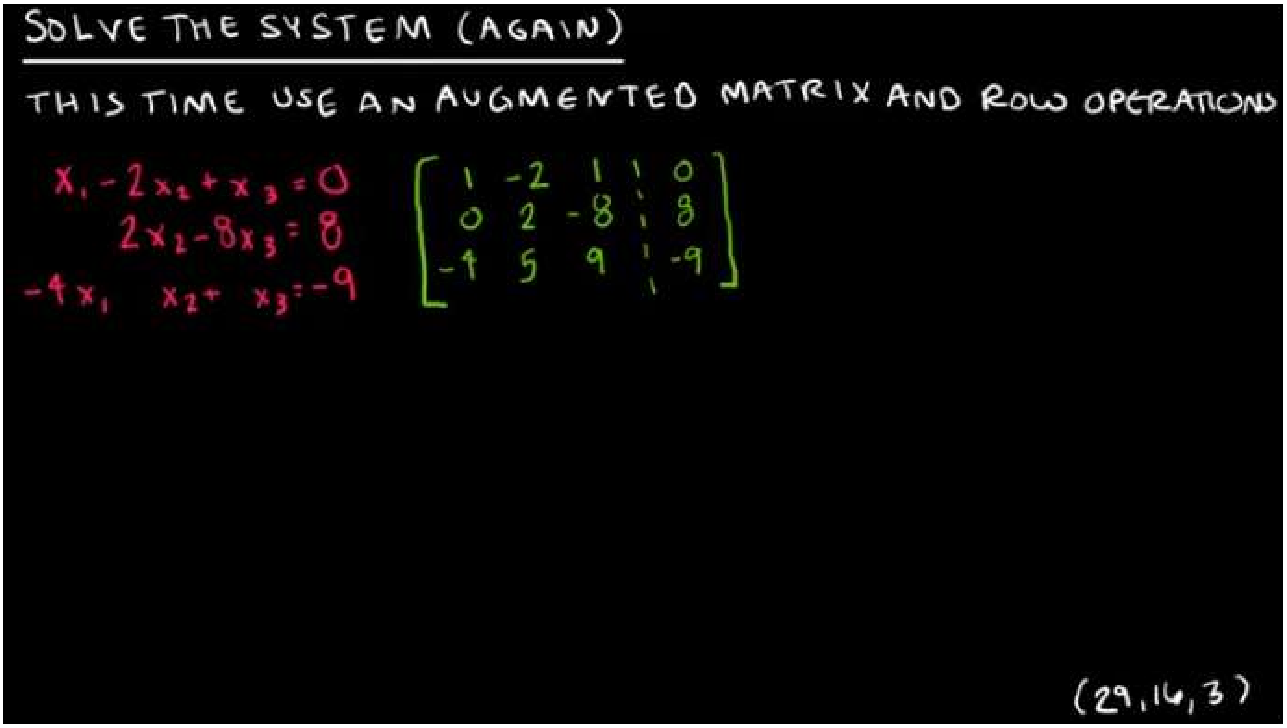
$-2 \quad 0 \quad -18$

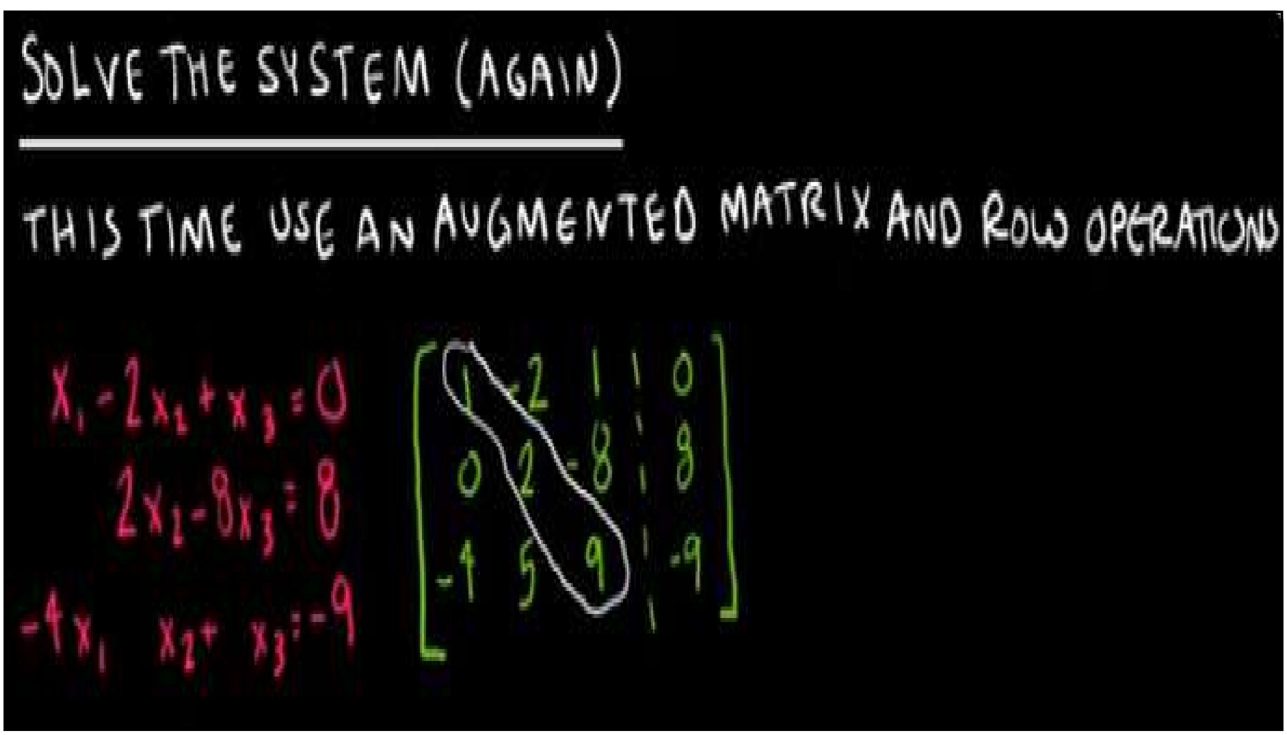
INTERCHANGE - INTERCHANGE (SWAP) TWO ROWS

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 2 & 4 & 8 \end{array}\right]$$

SCALING - MULTIPLY A ROW BY A NON-ZERO CONSTANT

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & 0 & 9 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 0 & 9 \end{array}\right]$$


SOLVE THE SYSTEM (AGAIN)
THIS TIME USE AN AUGMENTED MATRIX AND ROW OPERATIONS
$x_1 - 2x_2 + x_3 = 0$
$2x_2 - 8x_3 = 8$
$-4x_1 \quad x_2 + x_3 = -9$
$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$
(29, 16, 3)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$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$

SOLVE THE SYSTEM (AGAIN)

THIS TIME USE AN AUGMENTED MATRIX AND ROW OPERATIONS

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 \quad x_2 + x_3 = -9$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & \boxed{2} & -8 & 8 \\ \boxed{-4} & \boxed{5} & 9 & -9 \end{array}\right] \xrightarrow{4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Back Subst.

$$x_1 - 2x_2 + 1x_3 = 0$$
$$1x_2 - 4x_3 = 4$$
$$1x_3 = 3$$

$\rightarrow$ $x_2 - 4(3) = 4$, $x_2 - 12 = 4$, $x_2 = 16$

$x_1 - 2(16) + 3 = 0$, $x_1 - 32 + 3 = 0$, $x_1 - 29 = 0$, $x_1 = 29$

$(29, 16, 3)$

Try to complete these solutions using the row operations from here!

Hint? ...

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

EXISTENCE AND UNIQUENESS

DETERMINE IF THE SYSTEM IS CONSISTENT (DOES A SOLUTION EXIST?)

IF SO, DETERMINE IF THE SOLUTION IS UNIQUE (JUST ONE SOLUTION?)

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$4x_1 - 8x_2 + 12x_3 = 1$$

EXISTENCE AND UNIQUENESS

DETERMINE IF THE SYSTEM IS CONSISTENT (DOES A SOLUTION EXIST?)
IF SO, DETERMINE IF THE SOLUTION IS UNIQUE (JUST ONE SOLUTION?)

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

EXISTENCE AND UNIQUENESS

DETERMINE IF THE SYSTEM IS CONSISTENT (DOES A SOLUTION EXIST?)
IF SO, DETERMINE IF THE SOLUTION IS UNIQUE (JUST ONE SOLUTION?)

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ \boxed{4} & -8 & 12 & 1 \end{array}\right] \xrightarrow{-4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{array}\right] \xrightarrow{2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right]$$

$$0x_1 + 0x_2 + 0x_3 = 15$$

INCONSISTENT

LINEAR ALGEBRA

ROW REDUCTION AND ECHELON FORMS

ECHELON FORM VS. REDUCED ROW ECHELON FORM (RREF)

ECHELON: (PREVIOUSLY CALLED TRIANGLE FORM)

1) ALL NON-ZERO ROWS ARE ABOVE ALL ZERO ROWS
2) EACH LEADING ENTRY OF A ROW IS IN A COLUMN TO THE RIGHT OF THE LEADING ENTRY OF THE ROW ABOVE IT.
3) ALL ENTRIES IN A COLUMN BELOW A LEADING ENTRY ARE ZEROS.

RREF - ALL CONOITIONS ABOVE AND:

4) THE LEADING ENTRY IN EACH NON-ZERO ROW IS 1
5) EACH LEADING 1 IS THE ONLY NON-ZERO ENTRY IN THE COLUMN

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REF

$$\left[\begin{array}{cccc|c} \boxed{2} & 7 & 6 & 3 & 2 \\ 0 & \boxed{1} & 4 & 9 & 3 \\ 0 & 0 & 0 & \boxed{3} & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

PIVOT!

$$\left[\begin{array}{cccc|c} \boxed{1} & 4 & 2 & 3 & 9 \\ 0 & 0 & \boxed{2} & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

PIVOT POSITION - CORRESPONDS TO LEADING $\boxed{1}$ IN RREF

PIVOT COLUMN - THE COLUMN THAT CONTAINS THE PIVOT

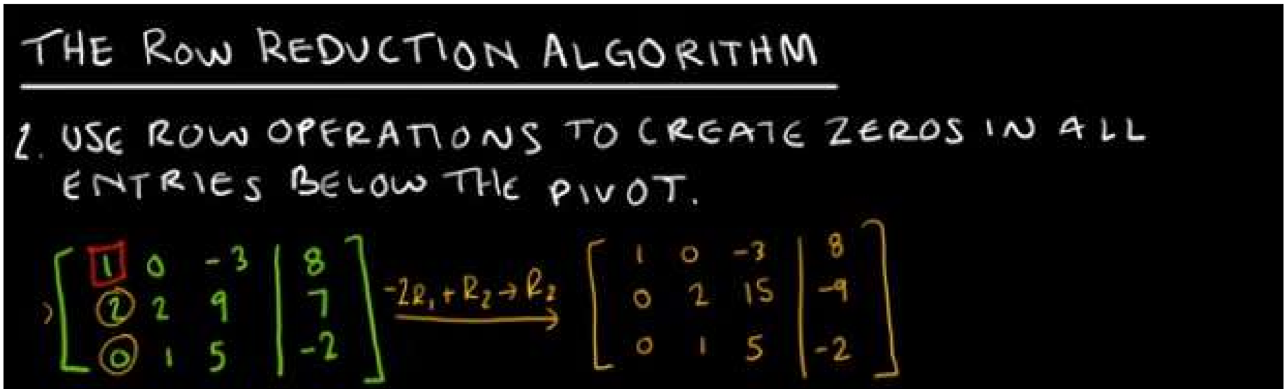
PIVOT - NONZERO NUMBER IN PIVOT POSITION USED TO CREATE ZEROS IN ROW OPERATIONS

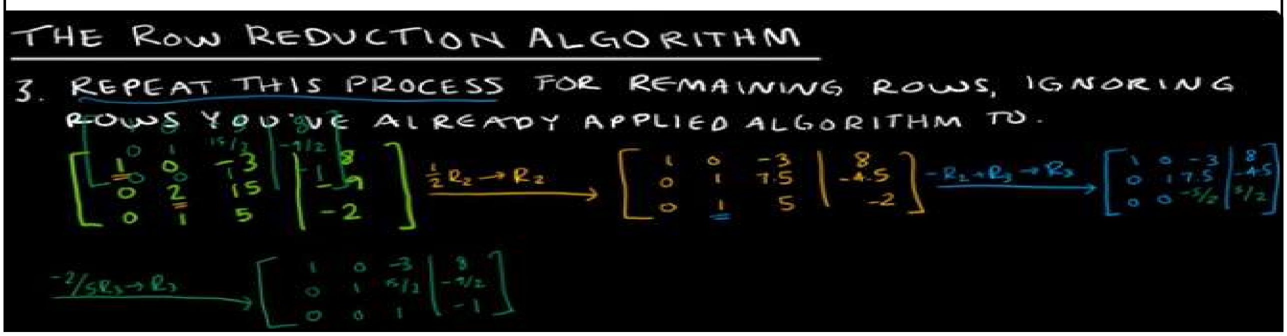
THE ROW REDUCTION ALGORITHM

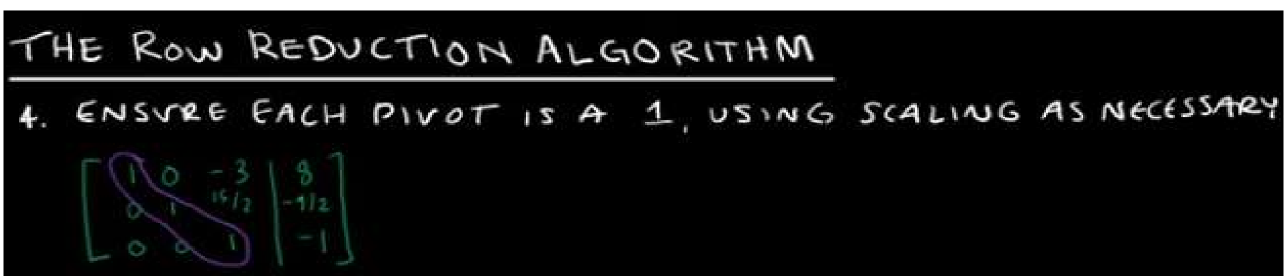
1. BEGIN AT LEFTMOST NONZERO COLUMN, WHICH IS A PIVOT COLUMN. SELECT A NONZERO ENTRY AS PIVOT AND INTERCHANGE, IF NECESSARY, TO MOVE THAT ENTRY INTO THE PIVOT POSITION (ROW 1).

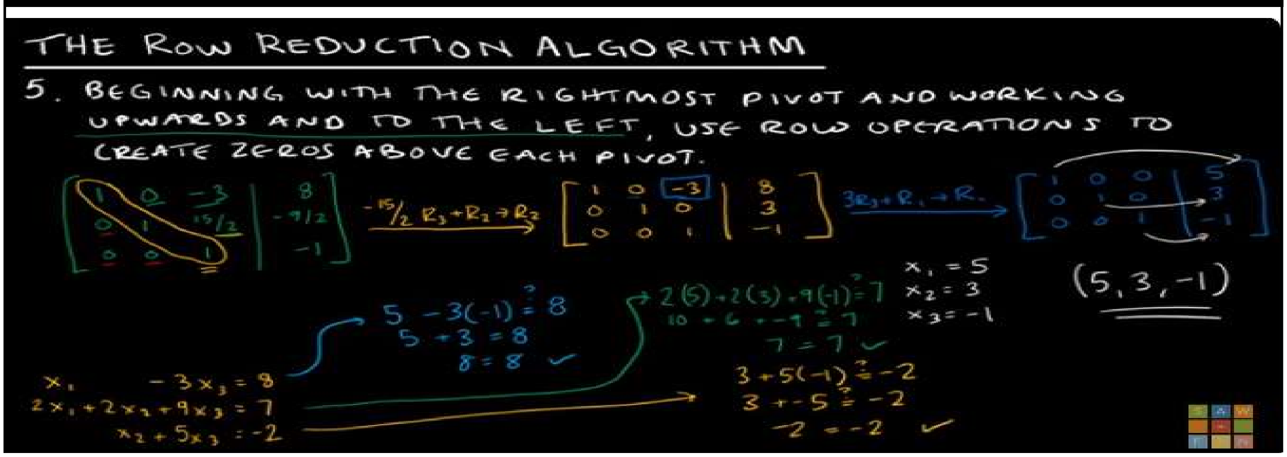
$$\begin{aligned} x_1 \qquad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$


THE ROW REDUCTION ALGORITHM
2. USE ROW OPERATIONS TO CREATE ZEROS IN ALL ENTRIES BELOW THE PIVOT.
-2R1+R2→R2


THE ROW REDUCTION ALGORITHM
3. REPEAT THIS PROCESS FOR REMAINING ROWS, IGNORING ROWS YOU'VE ALREADY APPLIED ALGORITHM TO.
½R2→R2
-R2+R3→R3
-2/5R3→R3


THE ROW REDUCTION ALGORITHM
4. ENSURE EACH PIVOT IS A 1, USING SCALING AS NECESSARY


THE ROW REDUCTION ALGORITHM
5. BEGINNING WITH THE RIGHTMOST PIVOT AND WORKING UPWARDS AND TO THE LEFT, USE ROW OPERATIONS TO CREATE ZEROS ABOVE EACH PIVOT.
-15/2 R3+R2→R2
3R3+R1→R1
x1 = 5
x2 = 3
x3 = -1
(5, 3, -1)
5 - 3(-1) ≟ 8
5 + 3 = 8
8 = 8 ✓
2(5)+2(3)+9(-1) ≟ 7
10 + 6 + -9 ≟ 7
7 = 7 ✓
3 + 5(-1) ≟ -2
3 + -5 ≟ -2
-2 = -2 ✓
x1 - 3x3 = 8
2x1 + 2x2 + 9x3 = 7
x2 + 5x3 = -2

LINEAR ALGEBRA

SOLUTION SETS AND FREE VARIABLES

CONSISTENT SYSTEM WITH INFINITELY MANY SOLUTIONS

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{r} x_1 \qquad - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{array} \begin{array}{l} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \left\{\begin{array}{l} x_1 = 5x_3 + 1 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{array}\right\}$$

FREE VARIABLES CAN TAKE ON ANY VALUE. ONCE YOU CHOOSE A VALUE FOR YOUR FREE VARIABLE, IT WILL DETERMINE THE VALUES OF THE OTHER (BASIC) VARIABLES.

LET $x_3 = 2$

$x_1 = 5(2) + 1 = 10 + 1 = 11$

$x_2 = 4 - 2 = 2$

$(11, 2, 2)$

LET $x_3 = -6$

$x_1 = 5(-6) + 1 = -30 + 1 = -29$

$x_2 = 4 + (+6) = 10$

$(-29, 10, -6)$

FIND THE GENERAL SOLUTION TO THE SYSTEM

$$x_1 - 2x_2 - x_3 + 3x_4 = 0$$
$$-2x_1 + 4x_2 + 5x_3 - 5x_4 = 3$$
$$3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2+R_3\to R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right]$$

$0 \neq 5$ INCONSISTENT

PRACTICE

FIND THE GENERAL SOLUTION OF THE AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & 4 & 7 \\ \boxed{3} & 9 & 7 & 6 \end{array}\right] \xrightarrow{-3R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 4 & 7 \\ 0 & 0 & \boxed{-5} & -15 \end{array}\right] \xrightarrow{-\frac{1}{5}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

★ $x_1 + 3x_2 + 4x_3 = 7$

$x_3 = 3$

$x_1 + 3x_2 + 4(3) = 7$

$x_1 + 3x_2 + 12 = 7$

$x_1 + 3x_2 = -5$

$x_1 = -5 - 3x_2$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

LINEAR ALGEBRA

VECTOR EQUATIONS

VECTORS IN $\mathbb{R}^2$

$(1,3) \rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

VECTOR - AN ORDERED LIST OF NUMBERS (MORE ON THIS IN CH. 4)

COLUMN VECTOR - A VECTOR WITH ONLY ONE COLUMN. WE OFTEN USE THESE FOR ORDERED PAIRS, TRIPLES, ETC.

VECTORS IN $\mathbb{R}^2$ - THE SET OF ALL VECTORS WITH 2 ENTRIES.
$\mathbb{R} \rightarrow$ REAL NUMBERS $2 \rightarrow$ NUMBER OF ENTRIES
THIS IS THE SET OF ALL POINTS IN A PLANE.

OPERATIONS WITH VECTORS - SAME AS WITH OTHER MATRICES.

$2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ SCALAR - MULTIPLY VECTOR BY A CONSTANT

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ADDITION - ADD CORRESPONING VALUES

MULTIPLICATION - NOPE! DIMENSIONS DON'T WORK.

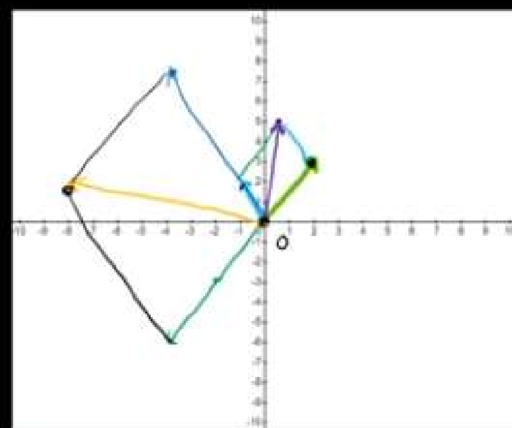
OPERATIONS ON VECTORS EXAMPLE

IF $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$ AND $\vec{v} = \begin{bmatrix}-1\\2\end{bmatrix}$, FIND $\vec{u}+\vec{v}$ AND $-2\vec{u}+4\vec{v}$.

$$\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

$$-2\begin{bmatrix}2\\3\end{bmatrix} + 4\begin{bmatrix}-1\\2\end{bmatrix}$$

$$\begin{bmatrix}-4\\-6\end{bmatrix} + \begin{bmatrix}-4\\8\end{bmatrix} = \begin{bmatrix}-8\\2\end{bmatrix}$$



PARALLELOGRAM RULE FOR ADDITION ★

VECTORS IN $\mathbb{R}^n$

IF $n \in \mathbb{R}$, THEN $\mathbb{R}^n$ IS THE COLLECTION OF ALL LISTS OF ORDERED N-TUPLES OF N REAL NUMBERS WRITTEN AS N × 1 COLUMN MATRICES.

$$\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\\\vdots\\u_n\end{bmatrix}$$

$$0 \quad \begin{bmatrix}0\\0\end{bmatrix}$$

ZERO VECTOR - THE VECTOR WHOSE ENTRIES ARE ALL 0, DENOTED BY 0.

$$\vec{v} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

THESE PROPERTIES CORRESPOND TO PROPERTIES OF REAL NUMBERS AND PERTAIN TO VECTORS u, v AND w AND SCALARS c AND d.

$\vec{u}+\vec{v}=\vec{v}+\vec{u}$

$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

$\vec{u}+0=0+\vec{u}=\vec{u}$

$\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=0$

$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$

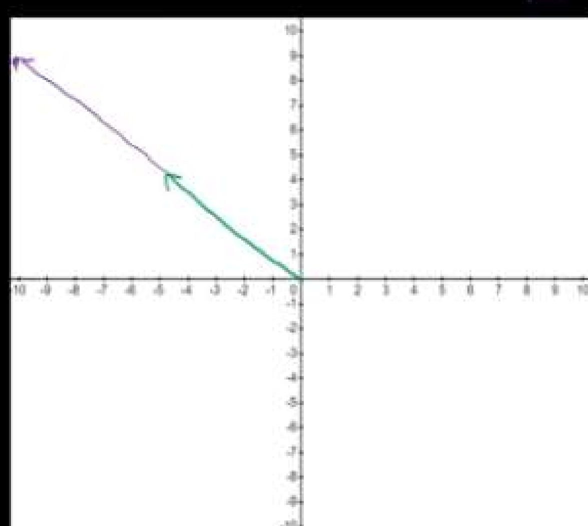
$(c+d)\vec{u}=c\vec{u}+d\vec{u}$

$c(d\vec{u})=(cd)\vec{u}$

$1\vec{u}=\vec{u}$

PRACTICE

LET $\vec{u}=\begin{bmatrix}-5\\4\end{bmatrix}$. DISPLAY THE VECTORS $\vec{u}$, $2\vec{u}$ AND $\frac{1}{2}\vec{u}$ ON A GRAPH.



SOME CLARIFICATION

A VECTOR EQUATION $\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}}$ HAS THE SAME SOLUTION SET AS THE LINEAR SYSTEM WHOSE AUGMENTED MATRIX IS $[\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n \mid \vec{b}]$. THEREFORE, A VECTOR EQUATION ONLY HAS A SOLUTION IF THE SYSTEM IS CONSISTENT.

IF $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p$ ARE IN $\mathbb{R}^n$, THEN THE SET OF LINEAR COMBINATIONS IS DENOTED SPAN$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ AND IS CALLED THE SUBSET OF $\mathbb{R}^n$ SPANNED. ESSENTIALLY SPAN$\{\vec{v}_1, \dots \vec{v}_p\}$ IS ALL VECTORS THAT CAN BE WRITTEN IN THE FORM $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = b$ WITH c_i SCALARS.

LINEAR ALGEBRA

LINEAR COMBINATIONS

LINEAR COMBINATIONS EXISTENCE

IF $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ AND $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, DETERMINE WHETHER $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ CAN BE WRITTEN AS A LINEAR COMBINATION OF v_1 AND v_2, THEN DETERMINE THE WEIGHTS SUCH THAT $c_1 v_1 + c_2 v_2 = b$.

$c_1 v_1 + c_2 v_2 = b$

$$\boxed{c_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} + c_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ -2c_1 \\ -5c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 5c_2 \\ 6c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \xrightarrow[5R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right] \xrightarrow{\frac{1}{9}R_2\to R_2} \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{array}\right] \xrightarrow[-16R_2+R_3\to R_3]{-2R_2+R_1\to R_1} \begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} v_1 & v_2 & b \end{array} \end{array}$$

$c_1 = 3$

$c_2 = 2$

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \checkmark$$

MORE SPAN TALK

NO

IS b IN THE PLANE CREATED BY SPAN $\{a_1, a_2\}$ IF $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ AND $\vec{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{array}\right] \xrightarrow{-6R_2+R_3\to R_3} \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{array}\right]$$

$0 \neq -2$

NO SOL.

PRACTICE

A MINING COMPANY HAS TWO MINES. ONE DAYS OPERATION AT MINE 1 PRODUCES ORE THAT CONTAINS 20 METRIC TONS OF COPPER AND 550 KG OF SILVER. MINE 2 PRODUCES 30 METRIC TONS OF COPPER AND 500 KG OF SILVER. HOW MANY DAYS SHOULD EACH MINE OPERATE TO PRODUCE 150 TONS COPPER AND 2825 KG SILVER.

$$x_1 \begin{bmatrix} 20 \\ 550 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 500 \end{bmatrix} = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1.5 & 7.5 \\ 550 & 500 & 2825 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1.5 & 7.5 \\ 0 & -325 & -1300 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1.5 & 7.5 \\ 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & 4 \end{array}\right]$$

MINE 1 1.5 days

MINE 2 4 days

PRACTICE 2

LET $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$ AND $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. FOR WHAT VALUE(S) OF h IS b IN THE SPAN OF a_1 AND a_2?

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & 8+h \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & 8+h \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 17+h \end{array}\right]$$

$0 = 0$

$17 + h = 0$

$h = -17$

$\boxed{\text{h must be -17 for b to be in the span } \{a_1, a_2\}}$

LINEAR ALGEBRA

THE MATRIX EQUATION $Ax = b$

LINEAR COMBINATIONS AS THE PRODUCT OF A MATRIX AND VECTOR

IF A IS AN $m \times n$ MATRIX WITH COLUMNS $a_1, a_2, \dots, a_n$ AND IF $x \in \mathbb{R}^n$, THEN Ax IS THE LINEAR COMBINATION OF THE COLUMNS OF A USING THE CORRESPONDING ENTRIES IN x AS WEIGHTS:

n!

we

$$A_x = \underset{1 \times n}{\overbrace{\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}}^{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \dots a_n x_n$$

EXAMPLES

1. FIND Ax IF $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ AND $x = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

$$Ax = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \begin{bmatrix} -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

2. WRITE THE LINEAR COMBINATION $\underline{2}v_1 - \underline{3}v_2 + \underline{4}v_3$ AS A MATRIX TIMES A VECTOR. Ax

$$Ax = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

THE SAME - BUT DIFFERENT

SYSTEM OF EQUATIONS:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 3 \\ 2x_2 + 3x_3 &= 4 \end{aligned}$$

AUGMENTED MATRIX:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ 0 & 2 & 3 & 4 \end{array}\right]$$

VECTOR EQUATION: $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$

$$x_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

MATRIX EQUATION: $Ax = b$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$